Domain, Range, & Symmetry

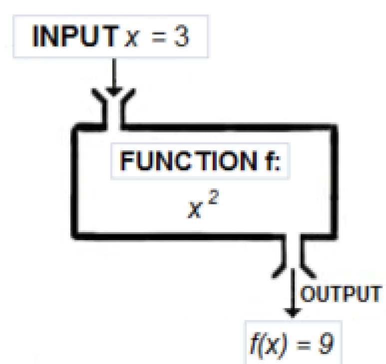
What happens when you put a log into a wood chipper? You get wood chips.

What happens when you put Styrofoam into a wood chipper? You get Styrofoam chips.

What happens if you put a metal rod into a wood chipper? You get a broken wood chipper.

Functions are "machines" much like a wood chipper. There are certain values that when put "into" the "machine" yield an output, and there are other values that "break" the machine and give no output.

The inputs are the independent variables, usually an x. The collection of all allowable inputs that yield an output is called the domain of the function.



The outputs are the dependent variables, usually a y or $f(x)$. The collection of all possible outputs generated from all the inputs is called the range of a function.

When studying functions or using a wood chipper, it's important to know what inputs are allowable and which are not. Once the domain is established, then the fun can begin of anticipating what the outputs will be.

We will be finding the domain of our functions analytically, that is, without seeing the graph of the function, while we will primarily be finding the range by sketching a graph and looking from low to high to see what y-values the graph takes on.

Finding Domains Analytically

When finding domains from a given function, there are only a handful of things that will restrict the domain.

1. Dividing by zero (just a handful of numbers)
2. Taking even roots of negative numbers (an infinite amount with a common trait)
3. Taking logarithms of non-positive numbers (an infinite amount with a common trait)
4. Explicitly undefined values, as in piecewise functions (can be a finite or infinite amount)

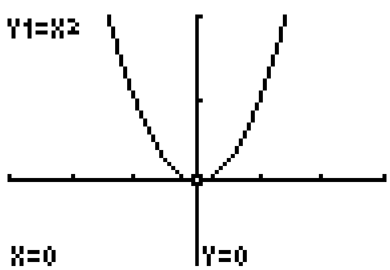
If you can safeguard against these four things from happening, you'll enjoy many years of care-free production from your function "machine." We'll look at an example of each real soon. But first . . .

Symmetry of graphs of functions
Knowing the existence or nonexistence of any symmetry a graph might have will help us sketch it more efficiently, give us insight into its behavior, and allow us to buy proper clothing for it when we're Christmas shopping.

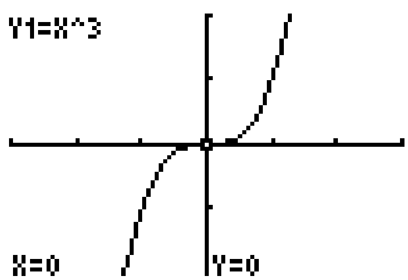
There are two main types of symmetry we're interested in, and functions exhibiting either of the two have special names.

1. *y*-axis symmetry (called **even** functions)
2. origin symmetry (called **odd** functions)

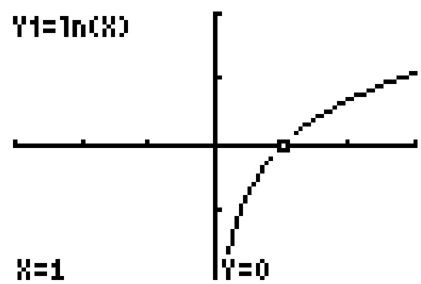
If we are looking at the visual representation of a function, looking for symmetry is an easy task. Graphs that lay on top of themselves exactly if they were folded along the *y*-axis have *y*-axis symmetry and are even functions. Graphs that look exactly the same when spun 180° ($\frac{\pi}{2}$ radians) around the origin have origin symmetry and are odd functions (notice "origin" and "odd" both start with the letter "o" and are mysteriously both missing the letter "z.")



y-axis symmetry
even function



origin symmetry
odd function

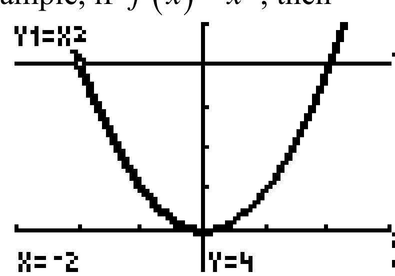


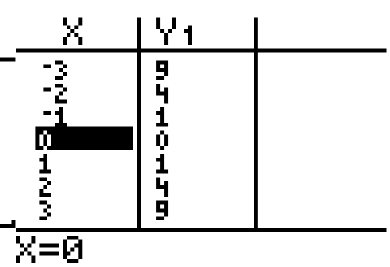
no symmetry
peculiar function

Wouldn't it be great to determine a function's symmetry (or lack thereof) from the equation alone? We're in luck. Understanding what's going on visually will enable us to develop a simple algebraic test for determining symmetry.

Even function
Because these functions have *y*-axis symmetry, they have identical *y*-values spaced evenly on either side of the *y*-axis for every *x*-value in the domain. For example, if $f(x)=x^2$, then

$$f(-2)=(-2)^2=4=(2)^2=f(2)$$
$$f(-\pi)=(-\pi)^2=\pi^2=(\pi)^2=f(\pi)$$
$$f(-100)=(-100)^2=10000=(100)^2=f(100)$$
$$f(-x)=(-x)^2=x^2=(x)^2=f(x)$$





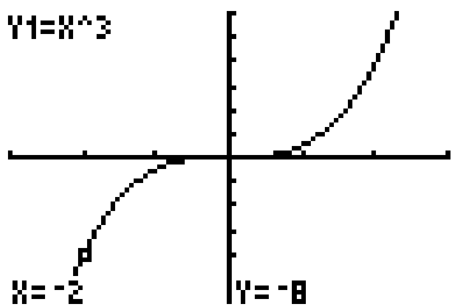
So in general, for ANY even function,

$$f(-x) = f(x), \text{ for all } x \in D_f$$

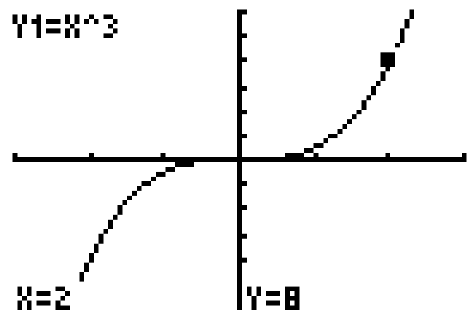
Algebraically, all you have to do is replace every x with $-x$ (carefully), simplify the expression, then compare it to the original function. If it's identical, you've found an even function. If it's *not* the same, it could be odd.

Odd function

Because these functions have origin symmetry, they have a rotational symmetry. Pieces of the graph, therefore, in quadrant I, where y is positive, must coincide with the graph in quadrant III, where y is negative. Similarly, quadrant II ($y > 0$) must be commensurate with the graphical features in quadrant IV ($y < 0$).



$$f(-2) = (-2)^3 = -8$$



$$f(2) = (2)^3 = 8$$

So

$$f(-2) = -f(2)$$

In general, for ANY odd function,

$$f(-x) = f(x), \text{ for all } x \in D_f$$

You're probably wondering if a function can be both even and odd and/or whether any function can have x-axis symmetry. The answer to both these questions is "YES!" There is one such function, and it is both even and odd and is the only function to possess x-axis symmetry. Do you know which function it is??

Example 1:

Find the domain of $f(x) = \dfrac{x}{3x^3 - 12x}$, then find and justify any symmetry.

Notice that we have a denominator with a variable in it. There might be a value or values that, when plugged into the function, yield a zero in the denominator. These values must be identified and "thrown out" of the prestigious domain club. In is generally easier to identify these values if the denominator is completely factored.

$$f(x) = \frac{x}{3x^3 - 12x}$$

$$f(x) = \frac{x}{3x\left(x^2 - 4\right)}$$

$$f(x) = \frac{x}{3x(x-2)(x+2)}$$

$x \neq 0,\ 2,\ -2$

so domain is $D_f : \{x | x \neq 0, \pm 2\}$

Test for symmetry:

$$f(-x) = \frac{(-x)}{3(-x)^3 - 12(-x)}$$

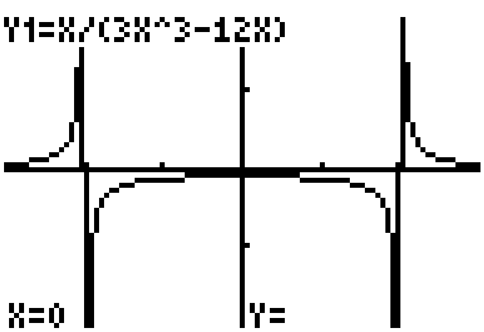
$$f(-x) = \frac{-x}{-3x^3 + 12x}$$

$$f(-x) = (-1)\left(\frac{x}{-3x^3 + 12x}\right) \neq -f(x)$$

$$f(-x) = \frac{-1}{-1}\left(\frac{x}{3x^3 - `12x}\right)$$

$$f(-x) = \frac{x}{3x^3 - `12x} = f(x)$$

So $f(x)$ is an even function.



- Most values that makes the denominator equal to zero are called **discontinuities** of the graph because the graph, existing on either side of the discontinuity, is interrupted by the singularity.
- There are two main types of discontinuities, infinite **non-removable** discontinuities (also known as Vertical Asymptotes or VAs) and **removable point** discontinuities (also known as holes).
- Notice that $x = 0$ also makes the numerator equal to zero yielding the indeterminate form of $\frac{0}{0}$. This means that the graph of $f(x)$ has a hole at $x = 0$. This happens also since the common factor of $\frac{x}{x}$ can be divided out, or *removed*, from the equation.
- The other two *x*-values of $x = \pm 2$ do NOT make the numerator equal to zero, yielding a $\frac{\neq 0}{0}$. This means that the graph of $f(x)$ has a vertical asymptote at each $x = -2$ and $x = 2$. Notice that the two factors of $(x-2)$ and $(x+2)$ CANNOT be divided out and are therefore *nonremovable*.
- Using the original, expanded version of the function is usually easier to use for symmetry tests.
- For a rational function, factor a negative out from either the numerator or denominator one at a time, then compare to the original function.
- Notice the function immediately did not appear to be even, but only after factoring out the second negative one from the denominator did our conclusion become obvious.

Example 2:

Find the domain of $f\left(x\right)=\frac{\sqrt{x-1}}{3x}$, then find and justify any symmetry.

Here we have a square root radical with a root index of two, an even number. Negatives under an even radical, that is, negative **radicands**, are mathematically impossible when talking about real numbers (and we are, by default, unless noted otherwise), since it's equally impossible to raise a negative number to an even exponent and obtain a negative number. Since there's a variable beneath the radical, there are likely many numbers that will give us a negative number, the type we want to exclude from the domain. We'll also have to guard against division by zero again, as there is a variable in the denominator. We'll focus on the radicand first.

$$f\left(x\right)=\frac{\sqrt{x-1}}{3x}$$
$$radicand \geq 0$$
$$x-1 \geq 0$$
$$x \geq 1$$

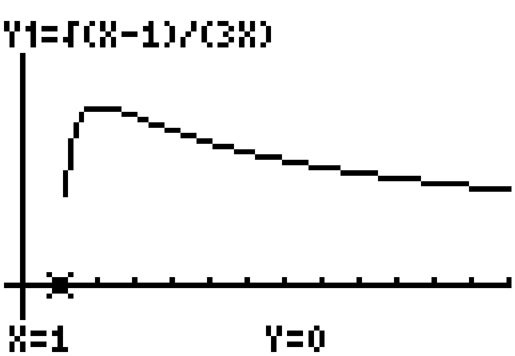
From the denominator, $x \neq 0$.

So $\boxed{D_f : \{x | x \geq 1\}}$

Test for symmetry:

$$f\left(-x\right)=\frac{\sqrt{(-x)-1}}{3(-x)}$$
$$f\left(-x\right)=\frac{\sqrt{-x-1}}{-3x}$$
$$f\left(-x\right)=\left(-1\right)\frac{\sqrt{-x-1}}{3x} \neq -f\left(x\right)$$
$$f\left(-x\right)=\left(-1\right)\frac{\sqrt{-(x+1)}}{3x} \neq f\left(x\right)$$

So f is neither even nor odd.



- For functions involving radicals, we can set up an inequality to find the values of x that DO work, that ARE in the domain by setting the radicand greater than or equal to zero.
- For functions involving division by zero, we usually find the few values that yield division by zero, then EXCLUDING those from the domain.
- In this example, the only value yielding division by zero was already excluded by our work with the radicand, so we don't need to throw it out again. If the division-by-zero value had not already been excluded, we would need to formally exclude it.
- When checking for symmetry, there's no way to pull the negative one out from under the radical since $\sqrt{-1}=i$, the imaginary unit, so we were unable to divide out the $\frac{-1}{-1}$ like in the previous example.
- Remember that calculators lie and graphs can be misleading. Notice that the graph doesn't show the y-values at or just to the right of $x=1$, even though they exist. It's because the graph is too steep, and your calculator eschews vertical lines in function mode. A quick numerical at $x=1$ confirms the function exists there.
- In function notation like $f\left(x\right)$, f is the name of the function, and x is the independent variable. When referring to a particular function, you refer to it by it name, f, or, if it's in trouble, it's full name, $f\left(x\right)$.

Example 3:

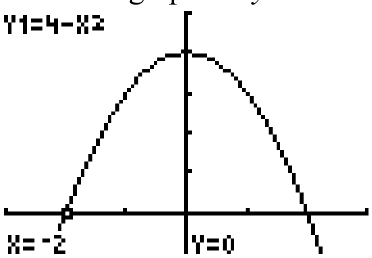
Find the domain of $g(t)=\dfrac{et}{(\pi t)\sqrt[4]{4-t^2}}$, then find and justify any symmetry.

Here we have a double whammy:
A variable in the denominator, under an even-root radical.

$radicand > 0$

$4-t^2>0$

From the graph of $y=4-t^2$



$-2<t<2$.

From the denominator, $t \neq 0$ so,

$\boxed{D_t:\{t \mid -2<t<2,\ t\neq 0\}}$

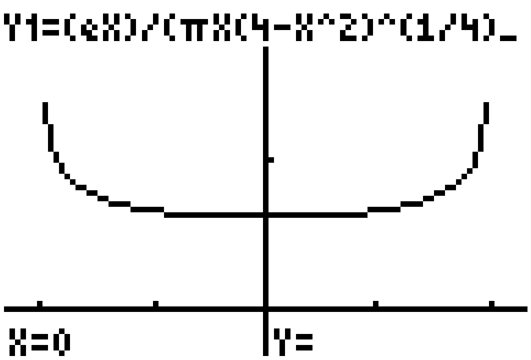
Test for symmetry:

$$g(-t)=\frac{e(-t)}{\pi(-t)\sqrt[4]{4-(-t)^2}}$$

$$g(-t)=\frac{\cancel{(-1)}et}{\cancel{(-1)}(\pi t)\sqrt{4-t^2}}$$

$$g(-t)=\frac{et}{(\pi t)\sqrt{4-t^2}}=g(t)$$

So *g* is an even function.



- Our function's name here is L'il *g*. The independent variable is *t*.
- We cannot solve the inequality algebraically in this case since the radicand is not linear. Instead we solve it graphically (from our knowledge of parent functions and transformations, or by finding the zeros of the radicand and testing the intervals between (we'll do this later in the year when finding relative extrema).
- Our inequality in this problem is strictly greater than, rather than greater than or equal. Although we can still take the fourth-root of zero, because it would give us a zero in the denominator, we want to omit this single value.
- The other value that yields division by zero other than $t=\pm 2$ is $t=0$. Because this value lies within our accepted domain interval, we need to exclude this singular value.
- When drawing *g* by hand, we would put open circles at $x=-2$, $x=0$, and $x=2$.
- Since the flow of the graph from left to right is interrupted at $x=0$, *g* has a discontinuity (a hole) at $x=0$.
- Technically we don't call $x=-2$ and $x=2$ discontinuities, since the flow of *g* was not interrupted. This is actually where the graph of *g* starts and stops, respectively (and non-inclusively).

Example 4:

Find the domain of $f(x)=\dfrac{\ln(x+3)}{\sqrt{4+x^2}}$

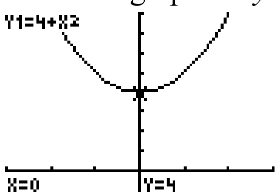
Here we have a logarithm and an even radical.

For the radical :

$radicand > 0$

$4+x^2>0$

From the graph of $y=4+x^2$



$4+x^2>0,\ \forall x\in\mathbb{R}$

From the logarithm:

$x+3>0$

$x>-3$

So $D_f:\{x|x>-3\}$

Or $D_f:(-3,\infty)$

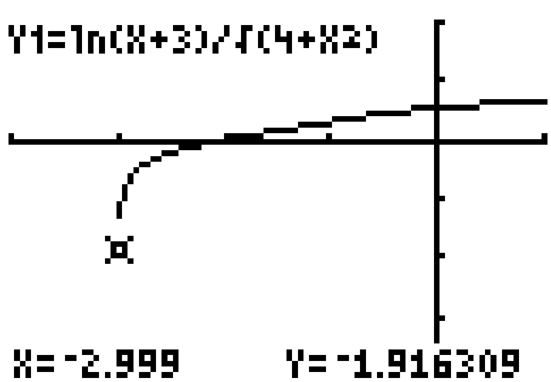
Test for symmetry:

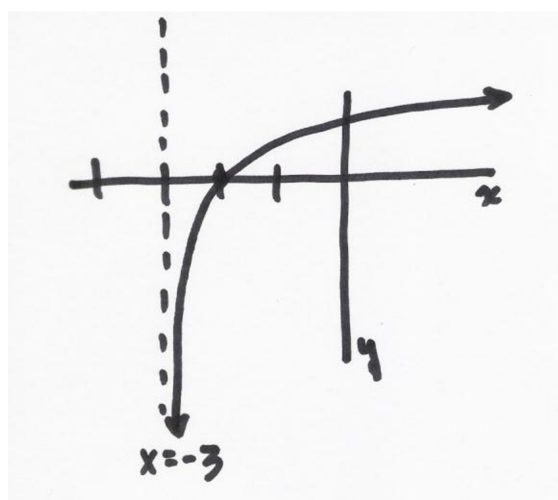
$f(-x)\neq -f(x)$

$f(-x)\neq f(x)$

So f is neither even nor odd.

- Just because a function has a radical, even an even radical, doesn't mean the radical will have domain restrictions.
- The "thing we take the log of" is called the **argument** of the logarithm.
- To algebraically find the domain of a log function, set the argument greater than zero, then solve the inequality.
- If the logarithm is in the denominator, you must also set the argument equal to one, solve, then exclude this value. This is because $\log_b 1 = 0$ for any base b.
- We can write our domain in either the traditional **set-builder notation** (as we have been using) or in the sometimes-more-convenient **interval notation**.
- In interval notation, a $\geq$ and $\leq$ are equivalent to a $[$ and a $]$ respectively.
- In interval notation, a $>$ and $<$ are equivalent to a $($ and $)$ respectively.
- In interval notation, we must always list from least to greatest, with a starting value and ending value separated by a comma.
- In interval notation, a ∞ or $-\infty$ ALWAYS gets a $($ or $)$.
- The infinity symbol, ∞, is called a **lemniscate**.
- The graph of $f(x)$ has a vertical asymptote at the root of the argument, here at $x=-3$. Notice again how the calculator's graph stops abruptly, even though the graph continues down to negative infinity as *x* approaches negative three from the right-hand side.





Example 5:

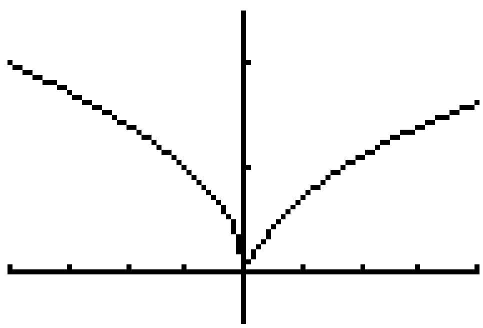
Find the domain of $f(x)=\begin{cases}\sqrt{-x}, & x\le 0\\ \ln(x+1), & x>0\end{cases}$, then find and justify any symmetry.

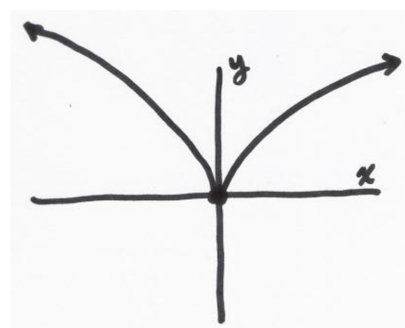
This is a piecewise function that is part $y=\sqrt{-x}$ and part $y=\ln(x+1)$. As the function indicates, the graph is $y=\sqrt{-x}$ to the left of and at $x=0$ and $y=\ln(x+1)$ to the right of $x=0$ ..

When finding the domain of a piecewise function, we should first look to see where the graph is explicitly defined. For this function, it is defined for all real numbers. Any domain restrictions, therefore, will have to come from the pieces on the intervals in which they're defined.

Knowing how to sketch parent functions will help you expedite the entire process, but you can find it analytically as well. In this case, each of the pieces is defined everywhere on its defined interval, so the domain is

$$D_f:\{x\,|\,x\in\mathbb{R}\}\ \text{ or }\ D_f:(-\infty,\infty)$$





$$f(x)=\begin{cases}\sqrt{-x}, & x\le 0\\ \ln(x+1), & x>0\end{cases}$$

Notice that the two pieces seem to meet at the origin, and in fact, they do! This means that either piece could lay claim to the function value at $x=0$ without changing the function's characteristics. However, because the equation is for a function, only one piece may be explicitly defined at $x=0$.

When it comes to symmetry, you have to be careful. At first glance, the graph itself looks like an even function, but as you already know, graphs can be misleading and calculators can lie. If you were sketching the graph without a calculator, you might actually draw it with *y*-axis symmetry. The algebraic analysis, however, clearly shows that this function is NOT an even function (nor is it odd).

$$f(-x)\ne f(x)\ \text{ and }\ f(-x)\ne -f(x)$$

This is a good time to mention the relative growth rate of each of the two functions making up the pieces of the graph above. Although they have similar shapes, the logarithmic function grows more slowly than the radical function. This is always the case: If fact, logarithmic functions are near the bottom of the "hierarchy of growth" charts, growing faster than only one function, the constant function (horizontal line) that doesn't grow at all! This will be valuable information once we start evaluating limits at infinity.

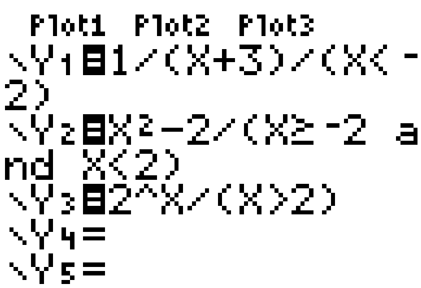
Example 6:

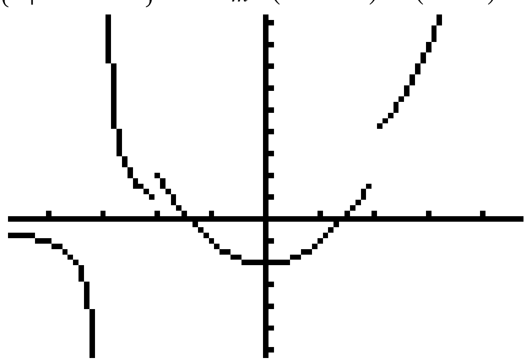
Find the domain of $m(z)=\begin{cases}\dfrac{1}{z+3}, & z<-2\\ z^2-2, & -2\le z<2\\ 2^z, & z>2\end{cases}$, then sketch the function. Look for any symmetry.

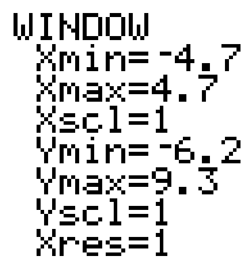
This is a 3-piece function. The most interesting *x*-values are $z=-2$ and $z=2$, the two values where the function changes from one piece to another. Careful inspection will reveal that the function is NOT defined at $z=2$, so $z=2$ is obviously not in the domain.

Now we look at the three pieces themselves. The quadratic piece (middle) and the exponential piece (bottom) themselves have domains of all real numbers, so we don't have to worry about them. The top piece, though, has a vertical asymptote at $z=-3$. Since we're using this piece everywhere left of $z=-2$, the vertical asymptote is part of the function.

The domain of *m* is therefore $D_m:\{z|z\neq -3,2\}$ or $D_m:(-\infty,-3)\cup(-3,2)\cup(2,\infty)$







*Notes:

- The $\cup$ is the "union" symbol, the mathematical symbol for the word "or." It is used to join disjoint intervals together.
- To graph a piecewise function, enter each piece in a separate line, and on each line, divide by the **Boolean argument**, that is the interval placed in parenthesis)
- Graphing a function in a "Zoom Decimal" window will often enable you see "holes" and eliminate the vertical lines where the asymptote lives. If you need a larger window, you can also multiply the *X*- or *Y*- scale by any scale factor to preserve this feature
- When sketching the graph by hand, you would want to put a solid dot and/or an open circle on the endpoints of each interval of the graph to show whether that point is inclusive or non-inclusive, as shown below.